

# Inverse Observability Inequalities for Integrodifferential Equations in Square Domains

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## Abstract

In this paper we will consider oscillations of square viscoelastic membranes by adding to the wave equation another term, which takes into account the memory. To this end, we will study a class of integrodifferential equations in square domains. By using accurate estimates of the spectral properties of the integrodifferential operator, we will prove an inverse observability inequality.

**Keywords:** observability; Fourier series; Ingham estimates

**MSC:** 45K05

## 1 Introduction

In [11] and [12] we solved a Dirichlet boundary control problem for the wave equation with the exponential memory kernel

$$k(t) = \beta e^{-\eta t}.$$

The result was established under some conditions on the parameters  $\beta$  and  $\eta$ , that is  $\eta > 3\beta/2$ ,  $\eta \geq 0$ ,  $\beta \geq 0$ . In the investigation a key point to get an estimate for the control time was to prove the inverse inequality. In [12] the analysis was done in the one-dimensional case, obtaining a precise estimate of the observability time  $T$ . In [13] we also solved the problem for  $n$ -dimensional balls. It was an open problem to extend to simple domains like rectangles, common in applications, the previous results using the Fourier method. The inverse observability estimate for the wave equation without memory was obtained under the geometrical condition that the control time  $T$  is greater than twice the diagonal of the rectangle, see [1]. By means of the Fourier method, Mehernberger [14] obtained a weaker result, nevertheless his method can be adapted to get the inverse inequality for other models. See also [7] for an improvement of [14] and further applications.

For another approach we have to mention the paper [5], where the analysis of the kernel is done by a compact perturbation and the author proves his result by means of a unique continuation property of the integro-differential equation. The method proposed by us is more direct and may be easily adapted to other boundary conditions, since in our estimates the dependence on the eigenvalues of the integro-differential operator is explicitly given. Moreover, Theorem 1.3 below has an interest in itself, because it contains a method which is more general than those used in [14, 7].

It is noteworthy that exponential kernels arise in viscoelasticity theory, such as in the analysis of Maxwell fluids or Poynting -Thomson solids, see e.g. [15, 17]. For other references in viscoelasticity theory see the seminal papers of Dafermos [2, 3] and [16, 9].

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In this paper we will consider oscillations of square viscoelastic membranes by adding to the wave equation another term, which takes into account the memory. We will fix  $\eta = 3\beta/2$  to study the integrodifferential equation in a square. This assumption has the double target to simplify the computation for the square and to extend to the 2-d case the results given in [12].

We will go back to the assumption  $\eta > 3\beta/2$  and observe that the estimates we need, still hold in the limiting case  $\eta = 3\beta/2$ . The analysis will require accurate estimates of the asymptotic behavior of the eigenvalues in the complex plane, with precise estimates for the limiting case.

We will consider the following Cauchy problem in the square domain  $\Omega = (0, \pi) \times (0, \pi)$

$$\begin{cases} u_{tt}(t, x, y) - \Delta u(t, x, y) + \beta \int_0^t e^{-\eta(t-s)} \Delta u(s, x, y) ds = 0, & t \in (0, T), (x, y) \in \Omega, \\ u(t, x, y) = 0, & t \in (0, T), (x, y) \in \partial\Omega, \\ u(0, x, y) = u_0(x, y), \quad u_t(0, x, y) = u_1(x, y), & (x, y) \in \Omega. \end{cases} \quad (1)$$

In [12] we provided a detailed analysis of the cubic equation associated with the integrodifferential equation. In particular, we gave the asymptotic behavior of the solutions of the cubic equation. Using those results, it is possible to write the solutions of the cubic equation in a different form with respect to that given in [12], but more fitting for the goal of the present paper. Indeed, the following representation for the solution of problem (1) holds.

**Theorem 1.1** *For any  $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$  and  $\eta \geq 3\beta/2$  the weak solution of problem (1) is given by*

$$u(t, x, y) = \sum_{k_1, k_2=1}^{\infty} \left( C_{k_1 k_2} e^{i\omega_{k_1 k_2} t} + \overline{C_{k_1 k_2}} e^{-i\overline{\omega_{k_1 k_2}} t} + R_{k_1 k_2} e^{r_{k_1 k_2} t} \right) \sin(k_1 x) \sin(k_2 y), \quad (2)$$

with

$$\begin{aligned} \Re \omega_{k_1 k_2} &= \sqrt{k_1^2 + k_2^2} (\Lambda_{k_1 k_2}^- + \Lambda_{k_1 k_2}^+), \\ \Im \omega_{k_1 k_2} &= \frac{1}{\sqrt{3}} \sqrt{k_1^2 + k_2^2} (\Lambda_{k_1 k_2}^- - \Lambda_{k_1 k_2}^+) + \frac{\eta}{3}, \\ r_{k_1 k_2} &= \frac{2}{\sqrt{3}} \sqrt{k_1^2 + k_2^2} (\Lambda_{k_1 k_2}^- - \Lambda_{k_1 k_2}^+) - \frac{\eta}{3}, \end{aligned} \quad (3)$$

where

$$\begin{aligned} \Lambda_{k_1 k_2}^- &= \frac{1}{2} \sqrt[3]{\Phi_{k_1 k_2} - \Psi_{k_1 k_2}}, \quad \Lambda_{k_1 k_2}^+ = \frac{1}{2} \sqrt[3]{\Phi_{k_1 k_2} + \Psi_{k_1 k_2}}, \\ \Phi_{k_1 k_2} &= \sqrt{1 + \left( 2\eta^2 + \frac{27\beta^2}{4} - 9\eta\beta \right) \frac{1}{k_1^2 + k_2^2} + \frac{\eta^3(\eta - \beta)}{(k_1^2 + k_2^2)^2}}, \\ \Psi_{k_1 k_2} &= \frac{\eta^3}{3\sqrt{3(k_1^2 + k_2^2)^3}} + \left( \eta - \frac{3\beta}{2} \right) \frac{\sqrt{3}}{\sqrt{k_1^2 + k_2^2}}. \end{aligned} \quad (4)$$

Moreover,

$$r_{k_1 k_2} \leq -\Im \omega_{k_1 k_2}, \quad \Im \omega_{k_1 k_2} \leq \frac{\eta}{3} \quad \forall k_1, k_2 \in \mathbb{N}, \quad (5)$$

and there exist  $\mu > 0$  such that

$$|R_{k_1 k_2}| \leq \mu \frac{|C_{k_1 k_2}|}{\sqrt{k_1^2 + k_2^2}} \quad \forall k_1, k_2 \in \mathbb{N}. \quad (6)$$

**Remark 1.2** *In formula (2) the coefficients  $C_{k_1 k_2}$  and  $R_{k_1 k_2}$  are uniquely determined by the initial conditions  $u_0$  and  $u_1$ . Since for our purposes it is only significant the relation (6) between  $R_{k_1 k_2}$  and  $C_{k_1 k_2}$ , we omit the explicit expression of  $R_{k_1 k_2}$ .*

In virtue of Theorem 1.1 we are able to establish the following observability estimate on the subset  $\Gamma = (0, \pi) \times \{0\} \cup \{0\} \times (0, \pi)$  of the boundary of the square domain.

**Theorem 1.3** *Let  $\eta = 3\beta/2$ . If  $u$  is the weak solution of problem (1) and  $\Gamma = (0, \pi) \times \{0\} \cup \{0\} \times (0, \pi)$ , then there exist  $\beta_0 > 0$  and  $T_0 > 0$  such that for all  $0 < \beta < \beta_0$  and  $T > T_0$  the inverse observability inequality*

$$\int_0^T \int_{\Gamma} \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma dt \geq c_0 \sum_{k_1, k_2=1}^{\infty} (k_1^2 + k_2^2) |C_{k_1 k_2}|^2 (1 + e^{-2\Im \omega_{k_1 k_2} T}), \quad (7)$$

holds true for some positive constant  $c_0 = c_0(T)$ .

We will prove Theorem 1.3 in Section 3.2 after some preliminary results.

## 2 Estimates of the eigenvalues

In this section we will study the distribution of the eigenvalues in the complex plane. Indeed, using the precise expressions of the eigenvalues provided by Theorem 1.1, we will analyze the behavior of partial gaps helpful to get the observability estimates.

To carry out our analysis, we need also the following known result, see [8].

**Lemma 2.1** *Fix an integer  $N \geq 2$  and  $N - 1$  integers  $k_1, \dots, k_{N-1} \geq 1$ . If  $k_N, k'_N$  are two positive integers satisfying*

$$\max \{k_N, k'_N\} \geq \max \{k_1, \dots, k_{N-1}\},$$

then

$$\left| \sqrt{k_1^2 + \dots + k_{N-1}^2 + k_N^2} - \sqrt{k_1^2 + \dots + k_{N-1}^2 + (k'_N)^2} \right| \geq (\sqrt{N} - \sqrt{N-1}) |k_N - k'_N|.$$

In particular, if  $N = 2$  one has  $\sqrt{N} - \sqrt{N-1} = \sqrt{2} - 1 \approx 0.41$ .

Using the notations introduced in Theorem 1.1 we will prove

**Proposition 2.2** *If  $\eta = 3\beta/2$  and  $\beta \in [0, \frac{2}{\sqrt{3}}]$ , there exists  $\gamma > 0$  such that*

$$\begin{aligned} |\Re \omega_{k_1 k_2} - \Re \omega_{k_1 k'_2}| &\geq \gamma |k_2 - k'_2| & \forall k_1 \in \mathbb{N}, \forall \max \{k_2, k'_2\} \geq k_1, \\ |\Re \omega_{k_1 k_2} - \Re \omega_{k'_1 k_2}| &\geq \gamma |k_1 - k'_1| & \forall k_2 \in \mathbb{N}, \forall \max \{k_1, k'_1\} \geq k_2, \end{aligned} \quad (8)$$

$$\Re \omega_{k_1 k_2} \geq \gamma \sqrt{k_1^2 + k_2^2}, \quad \forall k_1, k_2 \in \mathbb{N}. \quad (9)$$

Moreover, the constant  $\gamma = \gamma(\beta)$  in the previous inequalities can be taken equal to

$$\gamma = \frac{\sqrt{2}-1}{2} \left( \sqrt{1 - \frac{9}{8}\beta^2 + \frac{27}{64}\beta^4 + \frac{3\sqrt{3}}{16\sqrt{2}}\beta^3} \right)^{1/3} + \frac{\sqrt{2}-1}{2} \left( \sqrt{1 - \frac{9}{8}\beta^2 + \frac{27}{64}\beta^4 - \frac{3\sqrt{3}}{16\sqrt{2}}\beta^3} \right)^{1/3}. \quad (10)$$

In addition

$$0 \leq \Im \omega_{k_1 k_2} \leq \frac{\beta}{2} \quad \forall k_1, k_2 \in \mathbb{N}. \quad (11)$$

*Proof.* First, by taking  $\eta = \frac{3}{2}\beta$  in formulas (3) and (4), we obtain

$$\Re \omega_{k_1 k_2} = \sqrt{k_1^2 + k_2^2} (\Lambda_{k_1 k_2}^+(\beta) + \Lambda_{k_1 k_2}^-(\beta)). \quad (12)$$

where

$$\Phi_{k_1 k_2}(\beta) = \sqrt{1 - \frac{9}{4}\beta^2 \frac{1}{k_1^2 + k_2^2} + \frac{27}{16}\beta^4 \frac{1}{(k_1^2 + k_2^2)^2}}, \quad \Psi_{k_1 k_2}(\beta) = \frac{3\sqrt{3}}{8}\beta^3 \frac{1}{\sqrt{(k_1^2 + k_2^2)^3}}, \quad (13)$$

$$\begin{aligned} \Lambda_{k_1 k_2}^+(\beta) &= \frac{1}{2} \sqrt[3]{\Phi_{k_1 k_2}(\beta) + \Psi_{k_1 k_2}(\beta)} \\ &= \frac{1}{2} \left( \sqrt{1 - \frac{9}{4}\beta^2 \frac{1}{k_1^2 + k_2^2} + \frac{27}{16}\beta^4 \frac{1}{(k_1^2 + k_2^2)^2}} + \frac{3\sqrt{3}}{8}\beta^3 \frac{1}{\sqrt{(k_1^2 + k_2^2)^3}} \right)^{1/3}, \end{aligned} \quad (14)$$

$$\begin{aligned} \Lambda_{k_1 k_2}^-(\beta) &= \frac{1}{2} \sqrt[3]{\Phi_{k_1 k_2}(\beta) - \Psi_{k_1 k_2}(\beta)} \\ &= \frac{1}{2} \left( \sqrt{1 - \frac{9}{4}\beta^2 \frac{1}{k_1^2 + k_2^2} + \frac{27}{16}\beta^4 \frac{1}{(k_1^2 + k_2^2)^2}} - \frac{3\sqrt{3}}{8}\beta^3 \frac{1}{\sqrt{(k_1^2 + k_2^2)^3}} \right)^{1/3}. \end{aligned} \quad (15)$$

Fixed  $k_1 \in \mathbb{N}$  and  $k_2, k'_2 \in \mathbb{N}$  with  $k_2 > k'_2$ , thanks to (12) we have

$$\begin{aligned} \Re \omega_{k_1 k_2} - \Re \omega_{k_1 k'_2} &= \left( \sqrt{k_1^2 + k_2^2} - \sqrt{k_1^2 + (k'_2)^2} \right) \left( \Lambda_{k_1 k_2}^+(\beta) + \Lambda_{k_1 k_2}^-(\beta) \right) \\ &\quad + \sqrt{k_1^2 + (k'_2)^2} \left( \Lambda_{k_1 k_2}^+(\beta) + \Lambda_{k_1 k_2}^-(\beta) - \Lambda_{k_1 k'_2}^+(\beta) - \Lambda_{k_1 k'_2}^-(\beta) \right). \end{aligned} \quad (16)$$

We will show that the quantity  $\Lambda_{k_1 k_2}^+(\beta) + \Lambda_{k_1 k_2}^-(\beta)$ , regarded as function of  $\frac{1}{\sqrt{k_1^2 + k_2^2}}$ , is decreasing for  $\beta \in [0, \frac{2}{\sqrt{3}}]$ . To do that, in view of (14) and (15) we introduce the functions

$$F(x) = (f_+(x))^{1/3} + (f_-(x))^{1/3}, \quad f_{\pm}(x) = \sqrt{1 - x^2 + \frac{1}{3}x^4} \pm \frac{\sqrt{3}}{9}x^3, \quad (17)$$

since

$$\Lambda_{k_1 k_2}^{\pm}(\beta) = \frac{1}{2} f_{\pm} \left( \frac{3\beta}{2\sqrt{k_1^2 + k_2^2}} \right)^{1/3}, \quad \Lambda_{k_1 k_2}^+(\beta) + \Lambda_{k_1 k_2}^-(\beta) = \frac{1}{2} F \left( \frac{3\beta}{2\sqrt{k_1^2 + k_2^2}} \right). \quad (18)$$

We will prove that  $F(x)$  is decreasing in  $[0, \sqrt{\frac{3}{2}}]$ , that is  $F'(x) \leq 0$  for any  $x \in [0, \sqrt{\frac{3}{2}}]$ . First, we note that

$$F'(x) = \frac{1}{3} (f_+(x))^{-2/3} f'_+(x) + \frac{1}{3} (f_-(x))^{-2/3} f'_-(x).$$

Set  $a(x) = \sqrt{1 - x^2 + \frac{1}{3}x^4}$ , in view of

$$f'_{\pm}(x) = \frac{x}{3a(x)} (2x^2 - 3 \pm a(x)\sqrt{3}x) \quad x > 0,$$

we can write

$$F'(x) = \frac{x}{9a(x)} \left[ (f_+(x))^{-2/3} (2x^2 - 3 + a(x)\sqrt{3}x) + (f_-(x))^{-2/3} (2x^2 - 3 - a(x)\sqrt{3}x) \right].$$

Therefore,  $F'(x) \leq 0$  is equivalent to

$$\begin{aligned} (f_+(x))^{-2/3} (a(x)\sqrt{3}x + 2x^2 - 3) &\leq (f_-(x))^{-2/3} (a(x)\sqrt{3}x - (2x^2 - 3)), \\ (f_-(x))^2 (a(x)\sqrt{3}x + 2x^2 - 3)^3 &\leq (f_+(x))^2 (a(x)\sqrt{3}x - (2x^2 - 3))^3, \end{aligned}$$

and the last inequality is true for  $f_+(x) > f_-(x) > 0$  and  $2x^2 - 3 < 0$ , that is  $x \in [0, \sqrt{\frac{3}{2}}]$ . Therefore, it remains to be seen

$$f_-(x) > 0 \quad \forall x \in \left[0, \sqrt{\frac{3}{2}}\right]. \quad (19)$$

To this end, taking into account (17), we note that  $\sqrt{1 - x^2 + \frac{1}{3}x^4} > \frac{\sqrt{3}}{9}x^3$  if  $x^6 - 9x^4 + 27x^2 - 27 < 0$ . Because of  $x^6 - 9x^4 + 27x^2 - 27 = (x^2 - 3)^3$  we have that  $f_-(x) > 0$  for  $x < \sqrt{3}$ .

Since for  $\beta \in [0, \frac{2}{\sqrt{3}}]$  we have

$$\frac{3\beta}{2\sqrt{k_1^2 + k_2^2}} < \frac{3\beta}{2\sqrt{k_1^2 + (k'_2)^2}} \leq \frac{3\beta}{2\sqrt{2}} \leq \sqrt{\frac{3}{2}}, \quad (20)$$

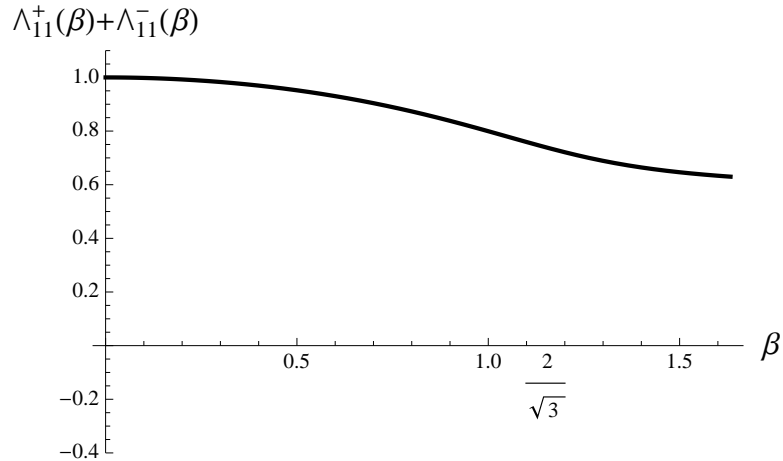
thanks to (18) we can deduce that

$$\Lambda_{k_1 k_2}^+(\beta) + \Lambda_{k_1 k_2}^-(\beta) \geq \Lambda_{11}^+(\beta) + \Lambda_{11}^-(\beta), \quad \Lambda_{k_1 k_2}^+(\beta) + \Lambda_{k_1 k_2}^-(\beta) \geq \Lambda_{k_1 k'_2}^+(\beta) + \Lambda_{k_1 k'_2}^-(\beta), \quad (21)$$

and hence, from (16) it follows

$$\Re \omega_{k_1 k_2} - \Re \omega_{k_1 k'_2} \geq (\Lambda_{11}^+(\beta) + \Lambda_{11}^-(\beta)) \left( \sqrt{k_1^2 + k_2^2} - \sqrt{k_1^2 + (k'_2)^2} \right). \quad (22)$$

Moreover, we also note that, thanks to  $f_+(x) > -f_-(x)$ , we have  $F(x) > 0$  for any  $x \in \mathbb{R}$ , whence we get  $\Lambda_{11}^+(\beta) + \Lambda_{11}^-(\beta) > 0$ .



Therefore from (22), using Lemma 2.1, we get

$$\Re \omega_{k_1 k_2} - \Re \omega_{k_1 k'_2} \geq (\sqrt{2} - 1)(\Lambda_{11}^+(\beta) + \Lambda_{11}^-(\beta))(k_2 - k'_2), \quad \forall k_1 < k_2,$$

so, we have shown the first inequality in (8) with  $\gamma = (\sqrt{2} - 1)(\Lambda_{11}^+(\beta) + \Lambda_{11}^-(\beta))$ , the same positive constant as in (10). In a similar way one can prove the other inequality in (8).

Finally, in virtue of (12) and (21) we obtain

$$\Re \omega_{k_1 k_2} \geq (\Lambda_{11}^+(\beta) + \Lambda_{11}^-(\beta)) \sqrt{k_1^2 + k_2^2}, \quad \forall k_1, k_2 \in \mathbb{N},$$

that is (9).

Regarding the last statement (11), first we note that  $\Im\omega_{k_1k_2} \leq \frac{\beta}{2}$  follows from (5) since  $\eta = 3\beta/2$ . To prove  $\Im\omega_{k_1k_2} \geq 0$ , we have to take  $\eta = \frac{3}{2}\beta$  in (3) and show

$$\Im\omega_{k_1k_2} = \frac{1}{\sqrt{3}}\sqrt{k_1^2 + k_2^2} (\Lambda_{k_1k_2}^-(\beta) - \Lambda_{k_1k_2}^+(\beta)) + \frac{\beta}{2} \geq 0,$$

that is

$$\frac{1}{\sqrt{3}}\sqrt{k_1^2 + k_2^2} (\Lambda_{k_1k_2}^+(\beta) - \Lambda_{k_1k_2}^-(\beta)) \leq \frac{\beta}{2}, \quad (23)$$

where  $\Lambda_{k_1k_2}^+(\beta)$  and  $\Lambda_{k_1k_2}^-(\beta)$  are given by (14) and (15) respectively. We observe that

$$\begin{aligned} & \Lambda_{k_1k_2}^+(\beta) - \Lambda_{k_1k_2}^-(\beta) \\ &= \frac{\Psi_{k_1k_2}(\beta)}{\sqrt[3]{(\Phi_{k_1k_2}(\beta) + \Psi_{k_1k_2}(\beta))^2} + \sqrt[3]{(\Phi_{k_1k_2}(\beta))^2 - (\Psi_{k_1k_2}(\beta))^2} + \sqrt[3]{(\Phi_{k_1k_2}(\beta) - \Psi_{k_1k_2}(\beta))^2}}. \end{aligned} \quad (24)$$

In virtue of (18), (19) and (20) we have

$$\sqrt[3]{\Phi_{k_1k_2}(\beta) - \Psi_{k_1k_2}(\beta)} = 2\Lambda_{k_1k_2}^-(\beta) = f_- \left( \frac{3\beta}{2\sqrt{k_1^2 + k_2^2}} \right)^{1/3} > 0.$$

Therefore, from (24) we get

$$\Lambda_{k_1k_2}^+(\beta) - \Lambda_{k_1k_2}^-(\beta) \leq \frac{\Psi_{k_1k_2}(\beta)}{\sqrt[3]{(\Phi_{k_1k_2}(\beta) + \Psi_{k_1k_2}(\beta))^2}},$$

whence, taking also into account (13), we have

$$\frac{1}{\sqrt{3}}\sqrt{k_1^2 + k_2^2} (\Lambda_{k_1k_2}^+(\beta) - \Lambda_{k_1k_2}^-(\beta)) \leq \frac{1}{\sqrt{3}}\sqrt{k_1^2 + k_2^2} \sqrt[3]{\Psi_{k_1k_2}(\beta)} = \frac{\beta}{2},$$

that is (23) holds true.  $\square$

**Remark 2.3** We observe that if we pass to the limit in (10) as  $\beta \rightarrow 0^+$ , we obtain  $\gamma = \sqrt{2} - 1$ , that is the value of the gap in the case of classical wave equations in a square domain, see Lemma 2.1.

### 3 The observability estimate

#### 3.1 The weight function

To prove our results we need to introduce the function

$$k(t) := \begin{cases} \sin \frac{\pi t}{T} & \text{if } t \in [0, T], \\ 0 & \text{otherwise.} \end{cases} \quad (25)$$

To begin with, we list some properties of  $k$  in the following lemma.

**Lemma 3.1** Set

$$K(u) := \frac{T\pi}{\pi^2 - T^2u^2}, \quad u \in \mathbb{C}, \quad (26)$$

the following properties hold for any  $u, z \in \mathbb{C}$

$$\overline{K(u)} = K(\bar{u}),$$

$$\int_{-\infty}^{\infty} k(t) \Re(z e^{iut}) dt = \Re(z(1 + e^{iuT})K(u)), \quad (27)$$

$$|K(u)| = |K(\bar{u})|. \quad (28)$$

For  $\gamma > 2\pi/T$ ,  $j \in \mathbb{N}$  and  $u \in \mathbb{C}$ ,  $|u| \geq \gamma j$ , we have

$$|K(u)| \leq \frac{4\pi}{T\gamma^2(4j^2 - 1)}. \quad (29)$$

*Proof.* We have to prove only the last statement. To this end, we observe that

$$|K(u)| = \frac{\pi}{T|u^2 - (\frac{\pi}{T})^2|} = \frac{4\pi}{T\gamma^2|4(\frac{u}{\gamma})^2 - (\frac{2\pi}{T\gamma})^2|}.$$

Since  $|u| \geq \gamma j$  and  $\frac{2\pi}{T\gamma} < 1$ , we have

$$\left|4\left(\frac{u}{\gamma}\right)^2 - \left(\frac{2\pi}{T\gamma}\right)^2\right| \geq 4\frac{|u|^2}{\gamma^2} - \left(\frac{2\pi}{T\gamma}\right)^2 \geq 4j^2 - 1,$$

and hence (29) follows.  $\square$

**Theorem 3.2** Assume that there exist  $\gamma > 0$  and  $\tau \in \mathbb{N}$  such that

$$|\Re\omega_n - \Re\omega_m| \geq \gamma|n - m| \quad \forall n, m \in \mathbb{N}, \max\{n, m\} \geq \tau, \quad (30)$$

and

$$\Re\omega_n \geq \gamma n \quad \forall n \in \mathbb{N}, \quad (31)$$

$$r_n \leq -\Im\omega_n \quad \forall n \in \mathbb{N}, \quad (32)$$

$$|R_n| \leq \mu \frac{|C_n|}{n^\theta} \quad \forall n \in \mathbb{N} \quad (\theta > 1/2, \mu > 0). \quad (33)$$

Then

$$\begin{aligned} \int_0^T \left| \sum_{n=1}^{\infty} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t}) \right|^2 dt \\ \geq 2T\pi \sum_{n=\tau}^{\infty} \left( \frac{1}{\pi^2 + 4T^2(\Im\omega_n)^2} - \frac{2S}{T^2\gamma^2} \right) |C_n|^2 (1 + e^{-2\Im\omega_n T}) \\ - \frac{8\pi}{T\gamma^2} \left( 1 + \frac{S}{2} \right) \sum_{n=1}^{\infty} |C_n|^2 (1 + e^{-2\Im\omega_n T}), \end{aligned} \quad (34)$$

where  $S = \mu \max\{\sum_{n=1}^{\infty} \frac{1}{n^{2\theta}}, \frac{\pi^2}{6}\}$ .

*Proof.* Set

$$F(t) = \sum_{n=1}^{\infty} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t} + R_n e^{r_n t}),$$

we note that

$$\begin{aligned} |F(t)|^2 &= \left| \sum_{n=\tau}^{\infty} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}) + \sum_{n=1}^{\tau-1} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}) + \sum_{n=1}^{\infty} R_n e^{r_n t} \right|^2 \\ &= \left| \sum_{n=\tau}^{\infty} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}) \right|^2 + 2 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\tau-1} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}) (\overline{C_m} e^{i\omega_m t} + C_m e^{-i\overline{\omega_m} t}) \\ &\quad + 2 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} R_m e^{r_m t} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}) + \left| \sum_{n=1}^{\tau-1} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega_n} t}) + \sum_{n=1}^{\infty} R_n e^{r_n t} \right|^2. \end{aligned}$$

Let  $k(t)$  be the function defined by (25). We have

$$\begin{aligned}
\int_{-\infty}^{\infty} k(t) |F(t)|^2 dt &= \int_{-\infty}^{\infty} k(t) \left| \sum_{n=\tau}^{\infty} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t}) \right|^2 dt \\
&+ 2 \int_{-\infty}^{\infty} k(t) \sum_{n=\tau}^{\infty} \sum_{m=1}^{\tau-1} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t}) (C_m e^{i\omega_m t} + \overline{C_m} e^{-i\overline{\omega}_m t}) dt \\
&+ 2 \int_{-\infty}^{\infty} k(t) \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} R_m e^{r_m t} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t}) dt \\
&+ \int_{-\infty}^{\infty} k(t) \left| \sum_{n=1}^{\tau-1} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t}) + \sum_{n=1}^{\infty} R_n e^{r_n t} \right|^2 dt.
\end{aligned}$$

Since  $k(t) \geq 0$  we can get rid of the last term on the right-hand side of the above formula, so we get

$$\begin{aligned}
\int_{-\infty}^{\infty} k(t) |F(t)|^2 dt &\geq \int_{-\infty}^{\infty} k(t) \left| \sum_{n=\tau}^{\infty} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t}) \right|^2 dt \\
&+ 2 \int_{-\infty}^{\infty} k(t) \sum_{n=\tau}^{\infty} \sum_{m=1}^{\tau-1} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t}) (C_m e^{i\omega_m t} + \overline{C_m} e^{-i\overline{\omega}_m t}) dt \\
&+ 2 \int_{-\infty}^{\infty} k(t) \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} R_m e^{r_m t} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t}) dt. \quad (35)
\end{aligned}$$

For  $n, m \in \mathbb{N}$  we have

$$\begin{aligned}
&(C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t}) (C_m e^{i\omega_m t} + \overline{C_m} e^{-i\overline{\omega}_m t}) \\
&= C_n C_m e^{i(\omega_n + \omega_m)t} + C_n \overline{C_m} e^{i(\omega_n - \overline{\omega}_m)t} + \overline{C_n} C_m e^{-i(\overline{\omega}_n - \omega_m)t} + \overline{C_n} \overline{C_m} e^{-i(\overline{\omega}_n + \overline{\omega}_m)t} \\
&= 2\Re(C_n \overline{C_m} e^{i(\omega_n - \overline{\omega}_m)t} + C_n C_m e^{i(\omega_n + \omega_m)t}),
\end{aligned}$$

so, by applying (27) we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} k(t) (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t}) (C_m e^{i\omega_m t} + \overline{C_m} e^{-i\overline{\omega}_m t}) dt \\
&= 2\Re(C_n \overline{C_m} (1 + e^{i(\omega_n - \overline{\omega}_m)T}) K(\omega_n - \overline{\omega}_m) + C_n C_m (1 + e^{i(\omega_n + \omega_m)T}) K(\omega_n + \omega_m)). \quad (36)
\end{aligned}$$

Similarly, by using again (27) we get

$$\begin{aligned}
&\int_{-\infty}^{\infty} k(t) e^{r_m t} (C_n e^{i\omega_n t} + \overline{C_n} e^{-i\overline{\omega}_n t}) dt = 2 \int_{-\infty}^{\infty} k(t) \Re(C_n e^{i(\omega_n - ir_m)t}) dt \\
&= 2\Re(C_n (1 + e^{(i\omega_n + r_m)T}) K(\omega_n - ir_m)). \quad (37)
\end{aligned}$$



By putting (36) and (37) into (35), we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} k(t) |F(t)|^2 dt \geq \\
& 2 \sum_{n,m=\tau}^{\infty} \Re \left( C_n \overline{C_m} (1 + e^{i(\omega_n - \overline{\omega_m})T}) K(\omega_n - \overline{\omega_m}) + C_n C_m (1 + e^{i(\omega_n + \omega_m)T}) K(\omega_n + \omega_m) \right) \\
& + 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\tau-1} \Re \left( C_n \overline{C_m} (1 + e^{i(\omega_n - \overline{\omega_m})T}) K(\omega_n - \overline{\omega_m}) + C_n C_m (1 + e^{i(\omega_n + \omega_m)T}) K(\omega_n + \omega_m) \right) \\
& + 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} R_m \Re \left( C_n (1 + e^{i(\omega_n + r_m)T}) K(\omega_n - ir_m) \right). \quad (38)
\end{aligned}$$

We may write the first sum on the right-hand side as follows

$$\begin{aligned}
& \sum_{n,m=\tau}^{\infty} \Re \left( C_n \overline{C_m} (1 + e^{i(\omega_n - \overline{\omega_m})T}) K(\omega_n - \overline{\omega_m}) \right) \\
& = \sum_{n=\tau}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}) K(\omega_n - \overline{\omega_n}) + \sum_{\substack{n,m=\tau \\ n \neq m}}^{\infty} \Re \left( C_n \overline{C_m} (1 + e^{i(\omega_n - \overline{\omega_m})T}) K(\omega_n - \overline{\omega_m}) \right).
\end{aligned}$$

Plugging the above identity into (38) we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} k(t) |F(t)|^2 dt \geq 2 \sum_{n=\tau}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}) K(\omega_n - \overline{\omega_n}) \\
& + 2 \sum_{\substack{n,m=\tau \\ n \neq m}}^{\infty} \Re \left( C_n \overline{C_m} (1 + e^{i(\omega_n - \overline{\omega_m})T}) K(\omega_n - \overline{\omega_m}) \right) + 2 \sum_{n,m=\tau}^{\infty} \Re \left( C_n C_m (1 + e^{i(\omega_n + \omega_m)T}) K(\omega_n + \omega_m) \right) \\
& + 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\tau-1} \Re \left( C_n \overline{C_m} (1 + e^{i(\omega_n - \overline{\omega_m})T}) K(\omega_n - \overline{\omega_m}) + C_n C_m (1 + e^{i(\omega_n + \omega_m)T}) K(\omega_n + \omega_m) \right) \\
& + 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} R_m \Re \left( C_n (1 + e^{i(\omega_n + r_m)T}) K(\omega_n - ir_m) \right).
\end{aligned}$$

By using the elementary estimate  $\Re z \geq -|z|$ ,  $z \in \mathbb{C}$ , we obtain

$$\begin{aligned}
& \int_{-\infty}^{\infty} k(t) |F(t)|^2 dt \geq 2 \sum_{n=\tau}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}) K(\omega_n - \overline{\omega_n}) \\
& - 2 \sum_{\substack{n,m=\tau \\ n \neq m}}^{\infty} |C_n| |C_m| (1 + e^{-(\Im \omega_n + \Im \omega_m)T}) |K(\omega_n - \overline{\omega_m})| - 2 \sum_{n,m=\tau}^{\infty} |C_n| |C_m| (1 + e^{-(\Im \omega_n + \Im \omega_m)T}) |K(\omega_n + \omega_m)| \\
& - 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\tau-1} |C_n| |C_m| (1 + e^{-(\Im \omega_n + \Im \omega_m)T}) \left( |K(\omega_n - \overline{\omega_m})| + |K(\omega_n + \omega_m)| \right) \\
& - 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} |C_n| |R_m| (1 + e^{(r_m - \Im \omega_n)T}) |K(\omega_n - ir_m)|. \quad (39)
\end{aligned}$$

By (28) we have  $|K(\omega_n - \overline{\omega_m})| = |K(\overline{\omega_n} - \omega_m)|$ , and hence

$$\sum_{\substack{n,m=\tau \\ n \neq m}}^{\infty} |C_n| |C_m| (1 + e^{-(\Im \omega_n + \Im \omega_m)T}) |K(\omega_n - \overline{\omega_m})| \leq \sum_{n=\tau}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}) \sum_{\substack{m=\tau \\ m \neq n}}^{\infty} |K(\omega_n - \overline{\omega_m})|. \quad (40)$$

Similarly

$$\sum_{n,m=\tau}^{\infty} |C_n| |C_m| (1 + e^{-(\Im \omega_n + \Im \omega_m)T}) |K(\omega_n + \omega_m)| \leq \sum_{n=\tau}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}) \sum_{m=\tau}^{\infty} |K(\omega_n + \omega_m)|. \quad (41)$$

Moreover,

$$\begin{aligned} \sum_{n=\tau}^{\infty} \sum_{m=1}^{\tau-1} |C_n| |C_m| (1 + e^{-(\Im \omega_n + \Im \omega_m)T}) & \left( |K(\omega_n - \overline{\omega_m})| + |K(\omega_n + \omega_m)| \right) \\ & \leq \frac{1}{2} \sum_{n=\tau}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}) \sum_{m=1}^{\tau-1} \left( |K(\omega_n - \overline{\omega_m})| + |K(\omega_n + \omega_m)| \right) \\ & \quad + \frac{1}{2} \sum_{m=1}^{\tau-1} |C_m|^2 (1 + e^{-2\Im \omega_m T}) \sum_{n=\tau}^{\infty} \left( |K(\omega_n - \overline{\omega_m})| + |K(\omega_n + \omega_m)| \right). \end{aligned} \quad (42)$$

Therefore, plugging formulas (40)–(42) into (39) we have

$$\begin{aligned} \int_{-\infty}^{\infty} k(t) |F(t)|^2 dt & \geq 2 \sum_{n=\tau}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}) |K(\omega_n - \overline{\omega_n})| \\ & \quad - 2 \sum_{n=\tau}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}) \left( \sum_{\substack{m=1 \\ m \neq n}}^{\infty} |K(\omega_n - \overline{\omega_m})| + \sum_{m=1}^{\infty} |K(\omega_n + \omega_m)| \right) \\ & \quad - 2 \sum_{n=1}^{\tau-1} |C_n|^2 (1 + e^{-2\Im \omega_n T}) \sum_{m=\tau}^{\infty} \left( |K(\omega_m - \overline{\omega_n})| + |K(\omega_m + \omega_n)| \right) \\ & \quad - 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} |C_n| |R_m| (1 + e^{(r_m - \Im \omega_n)T}) |K(\omega_n - ir_m)|. \end{aligned}$$

First, thanks to (26) we note that

$$K(\omega_n - \overline{\omega_n}) = \frac{T\pi}{\pi^2 + 4T^2(\Im \omega_n)^2},$$

so we can write

$$\begin{aligned} \int_{-\infty}^{\infty} k(t) |F(t)|^2 dt & \geq 2T\pi \sum_{n=\tau}^{\infty} |C_n|^2 \frac{1 + e^{-2\Im \omega_n T}}{\pi^2 + 4T^2(\Im \omega_n)^2} \\ & \quad - 2 \sum_{n=\tau}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}) \left( \sum_{\substack{m=1 \\ m \neq n}}^{\infty} |K(\omega_n - \overline{\omega_m})| + \sum_{m=1}^{\infty} |K(\omega_n + \omega_m)| \right) \\ & \quad - 2 \sum_{n=1}^{\tau-1} |C_n|^2 (1 + e^{-2\Im \omega_n T}) \sum_{m=\tau}^{\infty} \left( |K(\omega_m - \overline{\omega_n})| + |K(\omega_m + \omega_n)| \right) \\ & \quad - 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} |C_n| |R_m| (1 + e^{(r_m - \Im \omega_n)T}) |K(\omega_n - ir_m)|. \end{aligned} \quad (43)$$

Now, for any  $n \geq \tau$  we have to estimate the sum

$$\sum_{\substack{m=1 \\ m \neq n}}^{\infty} |K(\omega_n - \overline{\omega_m})| + \sum_{m=1}^{\infty} |K(\omega_n + \omega_m)|.$$

Since  $\max\{n, m\} \geq \tau$ , thanks to assumptions (30) and (31) we can apply (29) to get

$$\begin{aligned} \sum_{\substack{m=1 \\ m \neq n}}^{\infty} |K(\omega_n - \bar{\omega}_m)| + \sum_{m=1}^{\infty} |K(\omega_n + \omega_m)| \\ \leq \frac{4\pi}{T\gamma^2} \left( \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{1}{4(m-n)^2 - 1} + \sum_{m=1}^{\infty} \frac{1}{4(m+n)^2 - 1} \right) \leq \frac{8\pi}{T\gamma^2} \sum_{j=1}^{\infty} \frac{1}{4j^2 - 1}. \end{aligned}$$

Since  $\sum_{j=1}^{\infty} \frac{1}{4j^2 - 1} = \frac{1}{2}$ , we have

$$\sum_{\substack{m=1 \\ m \neq n}}^{\infty} |K(\omega_n - \bar{\omega}_m)| + \sum_{m=1}^{\infty} |K(\omega_n + \omega_m)| \leq \frac{4\pi}{T\gamma^2}.$$

In view of the above estimate, we can write (43) in the following way

$$\begin{aligned} \int_{-\infty}^{\infty} k(t) |F(t)|^2 dt &\geq 2T\pi \sum_{n=\tau}^{\infty} |C_n|^2 \frac{1 + e^{-2\Im\omega_n T}}{\pi^2 + 4T^2(\Im\omega_n)^2} - \frac{8\pi}{T\gamma^2} \sum_{n=\tau}^{\infty} |C_n|^2 (1 + e^{-2\Im\omega_n T}) \\ &\quad - 2 \sum_{n=1}^{\tau-1} |C_n|^2 (1 + e^{-2\Im\omega_n T}) \sum_{m=\tau}^{\infty} \left( |K(\omega_m - \bar{\omega}_n)| + |K(\omega_m + \omega_n)| \right) \\ &\quad - 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} |C_n| |R_m| (1 + e^{(r_m - \Im\omega_n)T}) |K(\omega_n - ir_m)|. \quad (44) \end{aligned}$$

Concerning the third sum on the right-hand side of the previous estimate, as above we can show that

$$\sum_{m=\tau}^{\infty} \left( |K(\omega_m - \bar{\omega}_n)| + |K(\omega_m + \omega_n)| \right) \leq \frac{4\pi}{T\gamma^2} \quad \forall n \leq \tau - 1,$$

and hence

$$\sum_{n=1}^{\tau-1} |C_n|^2 (1 + e^{-2\Im\omega_n T}) \sum_{m=\tau}^{\infty} \left( |K(\omega_m - \bar{\omega}_n)| + |K(\omega_m + \omega_n)| \right) \leq \frac{4\pi}{T\gamma^2} \sum_{n=1}^{\tau-1} |C_n|^2 (1 + e^{-2\Im\omega_n T}).$$

Therefore, from (44) it follows

$$\begin{aligned} \int_{-\infty}^{\infty} k(t) |F(t)|^2 dt &\geq 2T\pi \sum_{n=\tau}^{\infty} |C_n|^2 \frac{1 + e^{-2\Im\omega_n T}}{\pi^2 + 4T^2(\Im\omega_n)^2} - \frac{8\pi}{T\gamma^2} \sum_{n=1}^{\infty} |C_n|^2 (1 + e^{-2\Im\omega_n T}) \\ &\quad - 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} |C_n| |R_m| (1 + e^{(r_m - \Im\omega_n)T}) |K(\omega_n - ir_m)|. \quad (45) \end{aligned}$$

Now, we have to estimate the last sum on the right-hand side. Thanks to (33), we have

$$\begin{aligned} 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} |C_n| |R_m| |K(\omega_n - ir_m)| &\leq 4\mu \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} |C_n| \frac{|C_m|}{m^\theta} |K(\omega_n - ir_m)| \\ &\leq 2\mu \sum_{n=\tau}^{\infty} |C_n|^2 \sum_{m=1}^{\infty} \frac{|K(\omega_n - ir_m)|}{m^{2\theta}} + 2\mu \sum_{m=1}^{\infty} |C_m|^2 \sum_{n=\tau}^{\infty} |K(\omega_n - ir_m)|. \quad (46) \end{aligned}$$

Since  $\Re \omega_n \geq \gamma n$ , again by (29) we have

$$|K(\omega_n - ir_m)| \leq \frac{4\pi}{T\gamma^2(4n^2 - 1)}.$$

As a consequence, we get

$$\begin{aligned} 2\mu \sum_{n=\tau}^{\infty} |C_n|^2 \sum_{m=1}^{\infty} \frac{|K(\omega_n - ir_m)|}{m^{2\theta}} &\leq \frac{4\pi\mu}{T\gamma^2} \sum_{m=1}^{\infty} \frac{1}{m^{2\theta}} \sum_{n=\tau}^{\infty} \frac{|C_n|^2}{2n^2 - 1/2} \leq \frac{4\pi\mu}{T\gamma^2} \sum_{m=1}^{\infty} \frac{1}{m^{2\theta}} \sum_{n=\tau}^{\infty} |C_n|^2, \\ 2\mu \sum_{m=1}^{\infty} |C_m|^2 \sum_{n=\tau}^{\infty} |K(\omega_n - ir_m)| &\leq \frac{4\pi\mu}{T\gamma^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m=1}^{\infty} |C_m|^2. \end{aligned}$$

Plugging the two previous estimates into (46), we obtain

$$4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} |C_n| |R_m| |K(\omega_n - ir_m)| \leq \frac{4\pi\mu}{T\gamma^2} \sum_{n=1}^{\infty} \frac{1}{n^{2\theta}} \sum_{n=\tau}^{\infty} |C_n|^2 + \frac{4\pi\mu}{T\gamma^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{n=1}^{\infty} |C_n|^2.$$

In addition, keeping in mind also (32), in a similar way it follows

$$\begin{aligned} 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} |C_n| |R_m| e^{(r_m - \Im \omega_n)T} |K(\omega_n - ir_m)| &\leq 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} |R_m| e^{-\Im \omega_m T} |C_n| e^{-\Im \omega_n T} |K(\omega_n - ir_m)| \\ &\leq \frac{4\pi\mu}{T\gamma^2} \sum_{n=1}^{\infty} \frac{1}{n^{2\theta}} \sum_{n=\tau}^{\infty} |C_n|^2 e^{-2\Im \omega_n T} + \frac{4\pi\mu}{T\gamma^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{n=1}^{\infty} |C_n|^2 e^{-2\Im \omega_n T}, \end{aligned}$$

and hence, by summing the previous two inequalities yields

$$\begin{aligned} 4 \sum_{n=\tau}^{\infty} \sum_{m=1}^{\infty} |C_n| |R_m| (1 + e^{(r_m - \Im \omega_n)T}) |K(\omega_n - ir_m)| \\ \leq \frac{4\pi S}{T\gamma^2} \sum_{n=\tau}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}) + \frac{4\pi S}{T\gamma^2} \sum_{n=1}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}), \end{aligned}$$

where  $S = \mu \max\{\sum_{n=1}^{\infty} \frac{1}{n^{2\theta}}, \frac{\pi^2}{6}\}$ . Finally, putting the previous formula in (45), we get the estimate

$$\begin{aligned} \int_{-\infty}^{\infty} k(t) |F(t)|^2 dt &\geq 2T\pi \sum_{n=\tau}^{\infty} \left( \frac{1}{\pi^2 + 4T^2(\Im \omega_n)^2} - \frac{2S}{T^2\gamma^2} \right) |C_n|^2 (1 + e^{-2\Im \omega_n T}) \\ &\quad - \frac{8\pi}{T\gamma^2} \left( 1 + \frac{S}{2} \right) \sum_{n=1}^{\infty} |C_n|^2 (1 + e^{-2\Im \omega_n T}), \end{aligned}$$

whence, in virtue of the definition of  $k$ , we obtain (34)  $\square$

### 3.2 Proof of Theorem 1.3

Thanks to Theorem 1.1 the weak solution of problem (1) is given by

$$u(t, x, y) = \sum_{k_1, k_2=1}^{\infty} \left( C_{k_1 k_2} e^{i\omega_{k_1 k_2} t} + \overline{C_{k_1 k_2}} e^{-i\overline{\omega_{k_1 k_2}} t} + R_{k_1 k_2} e^{r_{k_1 k_2} t} \right) \sin(k_1 x) \sin(k_2 y).$$

Let  $\Gamma_1 = (0, \pi) \times \{0\}$ . We have

$$\begin{aligned}
\int_0^T \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma dt &= \int_0^T \int_0^\pi |u_y(t, x, 0)|^2 dx dt \\
&= \int_0^T \int_0^\pi \left| \sum_{k_1, k_2=1}^\infty k_2 \left( C_{k_1 k_2} e^{i\omega_{k_1 k_2} t} + \overline{C_{k_1 k_2}} e^{-i\overline{\omega_{k_1 k_2}} t} + R_{k_1 k_2} e^{r_{k_1 k_2} t} \right) \sin(k_1 x) \right|^2 dx dt \\
&= \frac{\pi}{2} \sum_{k_1=1}^\infty \int_0^T \left| \sum_{k_2=1}^\infty k_2 \left( C_{k_1 k_2} e^{i\omega_{k_1 k_2} t} + \overline{C_{k_1 k_2}} e^{-i\overline{\omega_{k_1 k_2}} t} + R_{k_1 k_2} e^{r_{k_1 k_2} t} \right) \right|^2 dt. \quad (47)
\end{aligned}$$

By Proposition 2.2 for any fixed  $k_1$  the assumptions of Theorem 3.2 are satisfied with  $\tau = k_1$ , so applying formula (34) we get

$$\begin{aligned}
&\int_0^T \left| \sum_{k_2=1}^\infty k_2 \left( C_{k_1 k_2} e^{i\omega_{k_1 k_2} t} + \overline{C_{k_1 k_2}} e^{-i\overline{\omega_{k_1 k_2}} t} + R_{k_1 k_2} e^{r_{k_1 k_2} t} \right) \right|^2 dt \\
&\geq 2T\pi \sum_{k_2=k_1}^\infty \left( \frac{1}{\pi^2 + 4T^2(\Im \omega_{k_1 k_2})^2} - \frac{2S}{T^2\gamma^2} \right) k_2^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im \omega_{k_1 k_2} T}) \\
&\quad - \frac{8\pi}{T\gamma^2} \left( 1 + \frac{S}{2} \right) \sum_{k_2=1}^\infty k_2^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im \omega_{k_1 k_2} T}),
\end{aligned}$$

where  $S = \frac{\pi^2}{6}\mu$ . The above formula can be also written in the following way

$$\begin{aligned}
&\int_0^T \left| \sum_{k_2=1}^\infty k_2 \left( C_{k_1 k_2} e^{i\omega_{k_1 k_2} t} + \overline{C_{k_1 k_2}} e^{-i\overline{\omega_{k_1 k_2}} t} + R_{k_1 k_2} e^{r_{k_1 k_2} t} \right) \right|^2 dt \\
&\geq 2T\pi \sum_{k_2=k_1}^\infty \frac{1}{\pi^2 + 4T^2(\Im \omega_{k_1 k_2})^2} k_2^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im \omega_{k_1 k_2} T}) \\
&\quad - \frac{8\pi}{T\gamma^2} (1 + S) \sum_{k_2=k_1}^\infty k_2^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im \omega_{k_1 k_2} T}) - \frac{8\pi}{T\gamma^2} \left( 1 + \frac{S}{2} \right) \sum_{k_2=1}^{k_1-1} k_2^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im \omega_{k_1 k_2} T}).
\end{aligned}$$

Now, we note that in the previous estimate we may change  $k_2^2$  into  $|k|^2 = k_1^2 + k_2^2$ . Indeed, if  $k_2 \geq k_1$  we have  $2k_2^2 \geq |k|^2$ , while for any  $k_1, k_2$  we have  $k_2^2 \leq |k|^2$ . So, thanks also to (11), we obtain

$$\begin{aligned}
&\int_0^T \left| \sum_{k_2=1}^\infty k_2 \left( C_{k_1 k_2} e^{i\omega_{k_1 k_2} t} + \overline{C_{k_1 k_2}} e^{-i\overline{\omega_{k_1 k_2}} t} + R_{k_1 k_2} e^{r_{k_1 k_2} t} \right) \right|^2 dt \\
&\geq \frac{T\pi}{\pi^2 + T^2\beta^2} \sum_{k_2=k_1}^\infty |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im \omega_{k_1 k_2} T}) \\
&\quad - \frac{8\pi}{T\gamma^2} (1 + S) \sum_{k_2=k_1}^\infty |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im \omega_{k_1 k_2} T}) - \frac{8\pi}{T\gamma^2} \left( 1 + \frac{S}{2} \right) \sum_{k_2=1}^{k_1-1} |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im \omega_{k_1 k_2} T}).
\end{aligned}$$

Therefore, in view of (47) it follows

$$\begin{aligned}
& \int_0^T \int_{\Gamma_1} \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma dt \\
& \geq \frac{T\pi^2}{2} \left( \frac{1}{\pi^2 + T^2\beta^2} - \frac{8}{T^2\gamma^2} (1+S) \right) \sum_{k_1=1}^{\infty} \sum_{k_2=k_1}^{\infty} |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im\omega_{k_1 k_2} T}) \\
& \quad - \frac{4\pi^2}{T\gamma^2} \left( 1 + \frac{S}{2} \right) \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1-1} |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im\omega_{k_1 k_2} T}). \quad (48)
\end{aligned}$$

In a similar way we can establish the following estimate for  $\Gamma_2 = \{0\} \times (0, \pi)$ :

$$\begin{aligned}
& \int_0^T \int_{\Gamma_2} \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma dt \\
& \geq \frac{T\pi^2}{2} \left( \frac{1}{\pi^2 + T^2\beta^2} - \frac{8}{T^2\gamma^2} (1+S) \right) \sum_{k_2=1}^{\infty} \sum_{k_1=k_2}^{\infty} |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im\omega_{k_1 k_2} T}) \\
& \quad - \frac{4\pi^2}{T\gamma^2} \left( 1 + \frac{S}{2} \right) \sum_{k_2=1}^{\infty} \sum_{k_1=1}^{k_2-1} |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im\omega_{k_1 k_2} T}) \\
& = \frac{T\pi^2}{2} \left( \frac{1}{\pi^2 + T^2\beta^2} - \frac{8}{T^2\gamma^2} (1+S) \right) \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1} |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im\omega_{k_1 k_2} T}) \\
& \quad - \frac{4\pi^2}{T\gamma^2} \left( 1 + \frac{S}{2} \right) \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+1}^{\infty} |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im\omega_{k_1 k_2} T}).
\end{aligned}$$

Thanks to the expression (10) of  $\gamma = \gamma(\beta)$ , there exists  $\beta_0 > 0$  such that  $\gamma^2 - 8(1+S)\beta^2 > 0$  for any  $\beta \in (0, \beta_0]$ , and hence, for  $T > 2\pi\sqrt{\frac{2(1+S)}{\gamma^2 - 8(1+S)\beta^2}}$  we get

$$\frac{1}{\pi^2 + T^2\beta^2} - \frac{8}{T^2\gamma^2} (1+S) = \frac{T^2(\gamma^2 - 8(1+S)\beta^2) - 8(1+S)\pi^2}{T^2\gamma^2(\pi^2 + T^2\beta^2)} > 0.$$

Therefore, from the previous estimates we can deduce

$$\begin{aligned}
& \int_0^T \int_{\Gamma_2} \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma dt \\
& \geq \frac{T\pi^2}{2} \left( \frac{1}{\pi^2 + T^2\beta^2} - \frac{8}{T^2\gamma^2} (1+S) \right) \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1-1} |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im\omega_{k_1 k_2} T}) \\
& \quad - \frac{4\pi^2}{T\gamma^2} \left( 1 + \frac{S}{2} \right) \sum_{k_1=1}^{\infty} \sum_{k_2=k_1}^{\infty} |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im\omega_{k_1 k_2} T}). \quad (49)
\end{aligned}$$

By summing (48) and (49) and taking into account that  $\Gamma = \Gamma_1 \cup \Gamma_2$  we get

$$\int_0^T \int_{\Gamma} \left| \frac{\partial u}{\partial \nu} \right|^2 d\Gamma dt \geq \frac{T\pi^2}{2} \left( \frac{1}{\pi^2 + T^2\beta^2} - \frac{4}{T^2\gamma^2} (4+3S) \right) \sum_{k_1, k_2=1}^{\infty} |k|^2 |C_{k_1 k_2}|^2 (1 + e^{-2\Im\omega_{k_1 k_2} T}).$$

Finally, again in view of (10), we can pick out  $\beta_0 > 0$  sufficiently small such that  $\gamma^2 - 4(4+3S)\beta^2 > 0$  for any  $\beta \in (0, \beta_0]$ , and hence

$$c_0 := \frac{T\pi^2}{2} \left( \frac{1}{\pi^2 + T^2\beta^2} - \frac{4}{T^2\gamma^2} (4+3S) \right) > 0 \quad \forall T > 2\pi\sqrt{\frac{4+3S}{\gamma^2 - 4(4+3S)\beta^2}},$$

so (7) holds true.  $\square$

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